Unfolding of Degenerate Hopf Bifurcation for Supersonic Flow past a Pitching Wedge

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This paper investigates the stability and bifurcation behavior of a double-wedge aerofoil performing a pitching motion at high angles of attack. When a pair of complex conjugate eigenvalues crosses the imaginary axis of the eigenvalue plane, the trivial solution loses stability giving rise to a periodic solution, known as Hopf bifurcation, provided certain transversality conditions are not violated. The existence of degenerate Hopf bifurcation due to the violation of Hopf's transversality condition at certain critical values of the system parameters is shown. The behavior of the pitching motion near these critical values is examined by unfolding the degeneracies. For the supersonic double-wedge aerofoil, various parameters defining the bifurcation paths were numerically evaluated.

I. Introduction

In recent years several new mathematical ideas have influenced the study of stability and bifurcation phenomena of nonlinear dynamical systems. In this paper, aerodynamic stability of a double-wedge subject to a single degree of freedom pitching motion is investigated. Recently Hui & Tobak[1] analyzed the Hopf bifurcation that results when a steady flight becomes unstable by increasing the angle of attack σ beyond a critical value σ_c, holding all other flow parameters fixed. If more than one parameter is allowed to vary, such as angle of attack σ and pivot position h, then phenomena other than simple Hopf bifurcation may occur. For the case of a double-wedge, it is found that if both angle of attack σ and pivot position h reach certain critical values σ_c and h_c, respectively, then the transversality condition of the Hopf bifurcation theorem does not hold and a so-called degenerate Hopf bifurcation takes place.[2] However, this degenerate phenomenon is nongeneric. In order to more completely understand the behavior of the system, it is useful to examine it near the singularities $\sigma=\sigma_c$ and $h=h_c$ by either incorporating an unfolding parameter or by studying the problem as a multiple parameter system.

In this paper, the former approach will be used to understand the bifurcation behavior of the system. A general framework for unfolding such degeneracies has been given by Golubitsky and Langford[3] using the singularity theory.

II. Statement of the Problem

Consider an aircraft in flight free to undergo a single degree of freedom pitching motion. The equations of pitching motion can be expressed as

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t}=\dot{\alpha}, \qquad I\frac{\mathrm{d}\dot{\alpha}}{\mathrm{d}t}=M(t) \tag{1}$$

where α is the angle of attack of the steady flight, I is the moment of inertia of the vehicle about the pivot axis, and $M(t)$ is the pitching moment at instantaneous time t of the aerodynamic forces about the same axis. When the motion is slowly varying,[4] the pitching moment $M(t)$ may be characterized with sufficient accuracy by the instantaneous angle of attack $\alpha(t)$ and the instantaneous rate of change of the angle of attack $\dot{\alpha}(t)$. Supose $\alpha=\sigma$ is an equilibrium state of the system of Eqs. (1); then, putting $\alpha(t)=\sigma+\psi(t)$, the variational equations about the equilibrium position can be written as

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}=\dot{\psi}$$

$$\frac{\mathrm{d}\dot{\psi}}{\mathrm{d}t}=\frac{M}{I}(t)=\frac{1}{2I}\rho_\infty V_\infty^2 \bar{S}L\,[C_m(0,0,\sigma,h)-C_m(\psi,\dot{\psi},\sigma,h)] \tag{2}$$

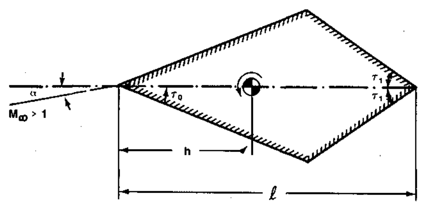

Fig. 1a Aerofoil at angle of attack.

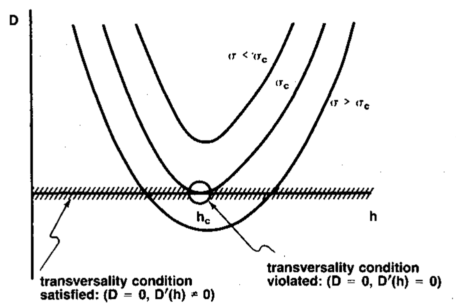

Fig. 1b Transversality condition and its violation.

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where ψ is the angular displacement of motion measured from the angle of attack σ of the steady flight; ρ_∞ and V_∞ are the freestream density and velocity, respectively; $\bar{S}$ and L are the reference area and length; and h represents the distance between the apex and the pivot position as defined in Fig. 1a. The function $C_m(\psi,\dot{\psi},\sigma,h)$ represents the pitching moment coefficient of the aerodynamic forces about the pivot axis and $C_m(0,0,\sigma,h)$ is its steady value at a fixed angle of attack σ. Even though C_m depends on the flight Mach number M_∞, the specific heats of the air and the aircraft shape, these parameters will be considered as "passive" parameters in this analysis. For a finite amplitude, slow, periodic, pitching motion with angular displacement $\psi(t)$ around a mean angle of attack σ, with terms of $0(\dot{\psi}^2,\ddot{\psi})$ assumed negligible, we can write[1]

$$-C_m(\psi,\dot{\psi},\sigma,h)=f(\sigma+\psi,h)+g(\sigma+\psi,h)\dot{\psi}$$

which reduces the second of Eqs. (2) to

$$\frac{d\dot{\psi}}{dt}=F(\psi,\dot{\psi},\sigma,h)$$

where

$$F(\psi,\dot{\psi},\sigma,h)=\frac{M(t)}{I}=\kappa[f(\sigma+\psi,h)-f(\sigma,h)$$

$$+g(\sigma+\psi,h)\dot{\psi}], \qquad \kappa=\frac{1}{2I}\rho_\infty V_\infty^2\bar{S}L$$

Equations (2) represent a pair of autonomous differential equations in R^2 the trivial solution of which is $\psi=0$. The objective of this investigation is to understand the stability of this trivial solution and the bifurcation behavior of Eqs. (2) as the system parameters σ and h are varied.

III. Stability of the Trivial Solution

The functions $f(\sigma,h)$ and $g(\sigma,h)$ are related to the stiffness derivative $S(\sigma,h)$ and the damping derivative $D(\sigma,h)$ of classical aerodynamics as follows:

$$\kappa\frac{\partial f}{\partial\sigma}(\sigma,h)=-S(\sigma,h), \qquad \kappa g(\sigma,h)=-D(\sigma,h)$$

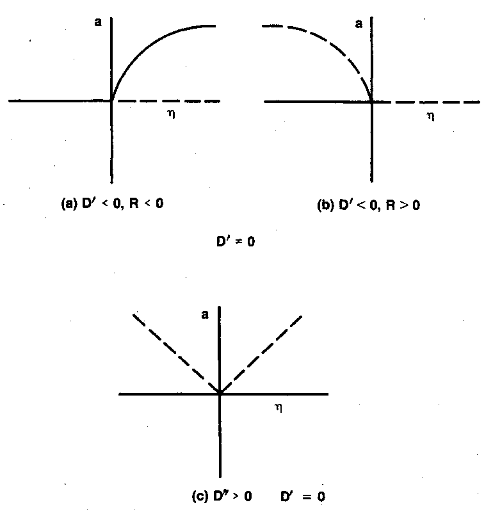

Fig. 2 Bifurcation diagrams: a) supercritical, b) subcritical, and c) degenerate Hopf bifurcation.

Introducing new state variables $y_1=\psi$, $y_2=\dot{\psi}$, Eqs. (2) may be written in the form

$$\dot{y}=Ay+\begin{bmatrix}0\\F(y_1,y_2,\sigma,h)\end{bmatrix}+0(|y|^4) \tag{3}$$

where

$$y=\begin{bmatrix}y_1\\y_2\end{bmatrix}$$

$$A(\sigma,h)=\begin{bmatrix}0 & 1\\-S(\sigma,h) & -D(\sigma,h)\end{bmatrix}$$

$$F(y_1,y_2,\sigma,h)=\bar{B}_{11}y_1^2+\bar{B}_{12}y_1y_2+\bar{C}_{111}y_1^3+\bar{C}_{112}y_1^2y_2$$

$$\bar{B}_{11}=-\frac{1}{2}\frac{\partial S}{\partial\sigma}(\sigma,h), \qquad \bar{B}_{12}=-\frac{1}{2}\frac{\partial D}{\partial\sigma}(\sigma,h)$$

$$\bar{C}_{111}=-\frac{1}{3!}\frac{\partial^2 S}{\partial\sigma^2}(\sigma,h), \qquad \bar{C}_{112}=-\frac{1}{2}\frac{\partial^2 D}{\partial\sigma^2}(\sigma,h)$$

The stability of the trivial solution is governed by the eigenvalues of the matrix A, which are

$$\lambda=-\frac{D}{2}\pm i\sqrt{S-D^2/4}=-\frac{D}{2}\pm i\omega \tag{4}$$

It is evident that the equilibrium position is asymptotically stable when

$$S(\sigma,h)>0, \qquad D(\sigma,h)>0$$

and instability occurs when $D(\sigma,h)=0$ and $S(\sigma,h)>0$, giving rise to a pair of pure imaginary eigenvalues; or when $D(\sigma,h)>0$ and $S(\sigma,h)=0$, giving rise to a zero and a negative eigenvalue; and the nongeneric case $D(\sigma,h)=0$ and $S(\sigma,h)=0$, giving rise to a double zero eigenvalue. Only the first case will be considered. Though the extension of the general results obtained in this paper for a two parameter system is possible, we shall analyze the problem as if it were a one parameter system. To avoid duplication of calculations, we shall refer to the bifurcation parameter as μ which can represent the angle of attack σ (or the pivot position h) holding h (or σ) constant. Let us assume that at $\mu=\mu_c$, the damping derivative becomes zero $D(\mu_c)=0$, the stiffness derivative $S(\mu_c)>0$, and the corresponding eigenvalues are $\lambda_c=\pm i$ and $\omega_c=\pm i\sqrt{S(\mu_c)}$. According to Hopf's theorem,[5] the system described by Eq. (3), along with the conditions

$$\omega(\mu_c)=\omega_0>0, \qquad D(\mu_c)=0 \tag{5a}$$

$$\left.\frac{dD}{d\mu}\right|_{\mu=\mu_\phi}=D'(\mu_c)\neq 0 \tag{5b}$$

has a family of periodic solutions bifurcating out of the equilibrium solution $y=0$, parameterized by the amplitude a for $|a|$ small. Furthermore, Hopf showed that along the periodic solution branch μ is an even function of a given by

$$\mu=\mu_c+\mu_2 a^2+\mu_4\alpha^4+\dots \tag{6a}$$

assuming

$$\mu_2\neq 0 \tag{6b}$$

These solutions exist either for $\mu>\mu_c$ (*supercritical* Hopf bifurcation) or for $\mu<\mu_c$ (*subcritical* Hopf bifurcation) depending on the sign of μ_2. Bifurcation of such periodic solutions out of the trivial solution, when Hopf's conditions, viz., Eqs. (5b) or (6b) or both Eqs. (5b) and (6b), are violated is in general called *degenerate* Hopf bifurcation. The preceding analysis holds for any single degree of freedom motion. Application of the analysis requires a knowledge of the stiffness derivative S and the damping derivative D together with their partial derivatives. The stiffness and damping derivatives for a double-wedge aerofoil in supersonic flow have been determined by Hui.[6] In this paper their partial derivatives have been calculated numerically using the results of Ref. 6. For the problem of double-wedge aerofoil it is the violation of Eq. (5b) which occurs, hence it is the degenerate Hopf bifurcation associated with the violation of Eq. (5b) which will be studied. Hopf's transversality condition, as well as its violation when h is taken as the bifurcation parameter, is shown in Fig. 1b.

IV. Bifurcation Analysis

In this section both Hopf and degenerate Hopf bifurcation will be considered. Assume that at $\mu=\mu_c$, the damping derivative becomes zero $[D(\mu_c)=0]$ and the stiffness derivative is positive $[S(\mu_c)>0]$. The eigenvalue is $\lambda_c=\pm i\omega_c=\pm i\sqrt{S(\mu_c)}$ and the corresponding eigenvector is $(1,\lambda_c)$.

To study the Hopf bifurcation and its stability, a change of coordinates is made to put the system of Eqs. (2) into a standard form. This is achieved by the linear transformation

$$y=Tx \tag{7}$$

where

$$T=\begin{bmatrix}1 & 0\\ 0 & \omega_c\end{bmatrix}$$

is the matrix consisting of the real and imaginary parts of the critical eigenvalue and $x=(x_1,x_2)$ represents the new state variables. The above transformation yields the system of equations with the linear part in standard form as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & \omega_c\\ -\omega_c & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}\dfrac{B_{11}}{\omega_c}x_1^2+B_{12}x_1x_2+\dfrac{{}^0C_{111}}{\omega_c}x_1^3+C_{112}x_1^2x_2\end{bmatrix} \tag{8}$$

Before proceeding to degenerate Hopf bifurcation, a summary of the results for the regular Hopf bifurcation will be given.

Hopf Bifurcation

Now the formulas for Hopf bifurcation given by Guckenheimer and Holmes[7] and Ariaratnam and Sri Namachchivaya,[2] are used to obtain the equation governing the bifurcating path

$$2Ra^3-D'(\mu_c)\eta a=0 \tag{9}$$

where

$$R=\frac{1}{8\omega_c^2}(B_{11}B_{12}+\omega_c^2C_{112}),\quad \eta=\mu-\mu_c\ll 1,$$

and a represents the amplitude of the bifurcating periodic solution given by

$$x_1=a\sin\phi+\frac{a^2}{\omega_c}\left[\frac{1}{2}\frac{B_{11}}{\omega_c}-\frac{1}{3}B_{12}\sin 2\phi+\frac{1}{6}\frac{B_{11}}{\omega_c}\cos 2\phi\right]$$

$$x_2=a\cos\phi-\frac{a^2}{\omega_c}\left[\frac{1}{3}\frac{B_{11}}{\omega_c}\sin 2\phi+\frac{2}{3}B_{12}\cos 2\phi\right] \tag{10}$$

where

$$\phi=\bar{\omega}t+\text{const}\qquad \bar{\omega}=\omega_c+a^2\left(P+\frac{\omega'}{D'}R\right)$$

$$P=\frac{1}{\omega_c^3}\left[\frac{1}{3}B_{11}^2-\frac{1}{6}\omega_c^2B_{12}^2-\frac{3}{8}\omega_c^2C_{111}\right]$$

The amplitude parameter relationship can be written using Eq. (9) as

$$\eta=\frac{2R}{D'(\mu_c)}a^2 \tag{11}$$

provided that $D'(\mu_c)\neq 0$.

When $D'(\mu)<0$, which is generally the case when eigenvalues cross from left to right in the complex λ-plane, it is evident from Eq. (11) that the bifurcating path exists for $\eta>0$ only if $R<0$ (*supercritical* bifurcation) as shown in Fig. 2a. Similarly the bifurcation path exists for $\eta<0$ only if $R>0$ (*subcritical* bifurcation) as shown in Fig. 2b. The opposite is true for $D'(\mu_c)>0$. It is well known that the damping and the stiffness derivatives are respectively quadratic and linear in h, i.e.,

$$D(\sigma,h)=D_0(\sigma)+D_1(\sigma)h+D_2(\sigma)h^2$$

$$S(\sigma,h)=S_0(\sigma)+S_1(\sigma)h \tag{12}$$

Furthermore, the qualitative variations of the quantities $D(\sigma,h)$ and $S(\sigma,h)$ with σ and h can be found in Hui[6] for double-wedge aerofoil. By considering σ as the bifurcation parameter, i.e., $\mu=\sigma$, the results of Hui and Tobak[1] are recovered. Similarly, the amplitude parameter relationship, considering h as the bifurcation parameter, can be written as

$$h-h_c=\frac{1}{8(D_1(\sigma)+2D_2(\sigma)h_c)}\left\{S\frac{\partial}{\partial\sigma}\left(\frac{\partial D/\partial\sigma}{S}\right)\right\}_{h=h_c} \tag{13}$$

Degenerate Hopf Bifurcation

Now we shall examine the bifurcations that can take place when Hopf's transversality condition [Eq. (5b)] is violated, i.e., degenerate Hopf bifurcation. It can be shown that in double-wedge and flat-plate aerofoils, degeneracies of the above-mentioned type for both parameters ($\partial D/\partial\sigma=0$, $\partial D/\partial h=0$) are present. However, $S(\sigma,h)>0$ only for the second case, and thus the degenerate Hopf bifurcation when $D=0$, $\partial D/\partial h=0$, will be examined, i.e., when

$$D_1^2(\sigma_c)=4D_0(\sigma_c)D_2(\sigma_c),\qquad h_c=-\frac{D_1(\sigma_c)}{2D_2(\sigma_c)}$$

provided $D_2(\sigma_c)\neq 0$. Violation of the transversality condition when h is considered as the bifurcation parameter is shown in Fig. 1b. Furthermore, for the eigenvalues to be purely imaginary we should have

$$S_0(\sigma_c)-\frac{S_1(\sigma_c)D_1(\sigma_c)}{2D_2(\sigma_c)}>0$$

Since we are studying the local behavior of the system, Eq. (8), as opposed to the global one, subsequent analysis is performed in small neighborhood of x, while the above conditions prevail. Thus, making use of the general results given in Ref. 5 for degenerate Hopf bifurcation, the equations governing the bifurcating path and improved frequency for the wedge problems are obtained as:

$$2Ra^3-D_2(\sigma_c)\bar{h}^2a=0 \tag{14a}$$

and

$$\bar{\omega}=\omega_c+a\left\{a\left[P-\frac{S_1^2(\sigma_c)R}{4\omega_c^2 D_2(\sigma_c)}\right]\pm\frac{S_1(\sigma_c)}{\omega_c}\left(\frac{R}{2D_2(\sigma_c)}\right)\right\} \quad (14b)$$

respectively, where $\bar{h}=h-h_c$. The existence of the bifurcating path depends on the sign of $R/D_2(\sigma_c)$. In other words, a bifurcating solution exists only if $R/D_2(\sigma_c)>0$. Therefore, in degenerate Hopf bifurcation, the bifurcating path exists on *both* sides of the a axis as opposed to Hopf bifurcation where the bifurcating path exists *either* for $\bar{h}>0$ or for $\bar{h}<0$. The bifurcating path can be expressed explicitly as

$$h-h_c=\pm\frac{a_0}{2\sqrt{D_2(\sigma)}}\left\{S\frac{\partial}{\partial\sigma}\left(\frac{\partial D/\partial\sigma}{S}\right)\right\}^{1/2}_{h=h_c} \quad (15)$$

It may be noted that each bifurcating path defined by Eq. (15) has a distinct frequency given by Eq. (14b). If $D_2>0$, then both bifurcating paths given in Eq. (15) are unstable while the trivial solution is stable. On the other hand, if $D_2<0$ no bifurcating solution exists. These results are shown in Fig. 2c.

V. Unfolding

Now to consider the behavior of the system near this nongeneric degenerate Hopf bifurcation, an unfolding parameter is introduced. Since the degeneracy occurs while considering h as a bifurcation parameter, it is natural to consider σ as an unfolding parameter. Loosely speaking, a parameter is said to be an unfolding parameter when it fills in the missing lower order term in the bifurcation equation. The main theoretical results classifying various bifurcations and their unfoldings when the conditions of Eq. (5b) or (6b) or both fail were presented by Golubitsky and Langford[3] using singularity theory. Making use of available results,[3] the equation governing the bifurcating paths incorporating $\partial D/\partial\sigma$ can be written as

$$2Ra^3-\left(\frac{\partial^2 D}{\partial h^2}\bigg|_{\sigma_c,h_c}\bar{h}^2+\frac{\partial D}{\partial\sigma}\bigg|_{\sigma_c,h_c}\bar{\sigma}\right)a=0 \quad (16)$$

which simplifies to

$$a^2-\frac{D_2(\sigma_c)}{R}\bar{h}^2-\beta=0 \quad (17)$$

where

$$\beta=\frac{1}{2R}[D_0'(\sigma_c)+D_1'(\sigma_c)h_c+D_2'(\sigma_c)h_c^2]\bar{\sigma}$$

$$\bar{\sigma}=\sigma-\sigma_c$$

$$\bar{h}=h-h_c$$

Depending on the sign of $D_2(\sigma_c)/R$ and β, a set of bifurcation diagrams as shown in Fig. 3 can be obtained. For $\beta=0$, Eq. (17) yields the previously obtained result of Eq. (15). In Fig. 3a the case $D_2(\sigma_c)/R<0$ is considered, for which Eq. (17) represents an ellipse for $\beta>0$ and has no real solution for $\beta<0$. On the other hand, for $D_2(\sigma_c)/R>0$ Eq. (17) represents an hyperbola for $\beta\neq 0$ as sketched in Fig. 3b where s and u indicate stable and unstable solutions, respectively.

Applying the results obtained in the above analysis to Hui's solution[6] for a double-wedge aerofoil, it is found that the case corresponding to Fig. 3b occurs. The special cases of a flat plate aerofoil and a wedge may be obtained from Hui's solution[6] by an appropriate choice of shape parameters τ_0, τ_1 in Fig. 1a. For the purposes of this study, we focus our attention on the case $\tau_0=\tau_1=5°$ since this approximates a thin aerofoil. For a double-wedge in supersonic flight the various components of the stiffness and damping derivatives, namely S_0, S_1, D_0, D_1, and D_2 are plotted in Figs. 4 and 5. Using Eq. (12), $S(\sigma,h)$ and $D(\sigma,h)$ for a given value of σ and h may be obtained.

In addition to these results, various other quantities required for the bifurcation analysis and unfolding are also calculated and displayed in Figs. 6 to 8. In Fig. 6 the relationships between h_c and σ_c and between M_∞ and σ_c are plotted. From this figure one may obtain the critical value of h and σ

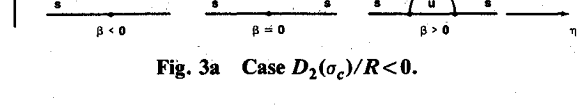

Fig. 3a Case $D_2(\sigma_c)/R<0$.

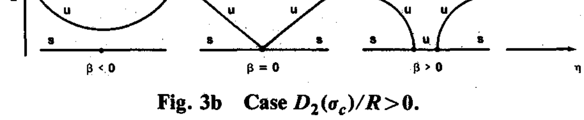

Fig. 3b Case $D_2(\sigma_c)/R>0$.

Fig. 3 Unfoldings.

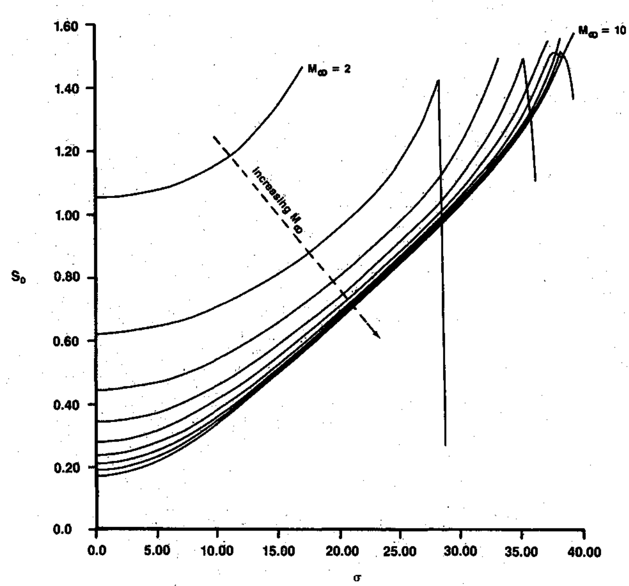

Fig. 4a S_0 vs σ for $M_\infty=2,3,\ldots,10$.

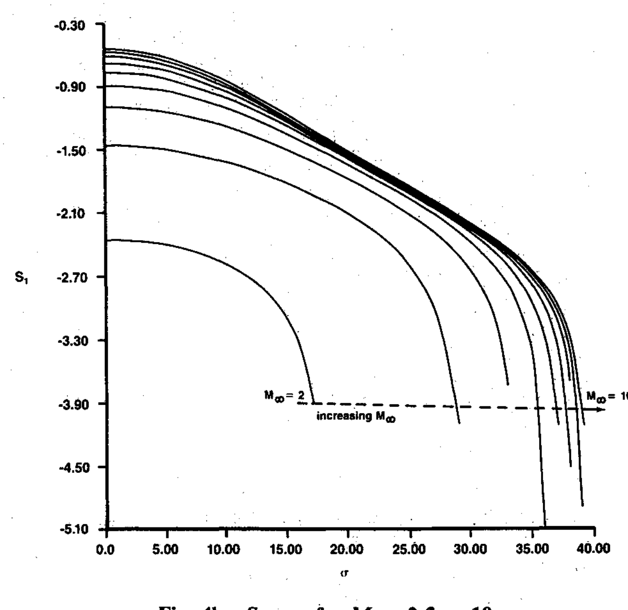

Fig. 4b S_1 vs σ for $M_\infty=2,3,\ldots,10$.

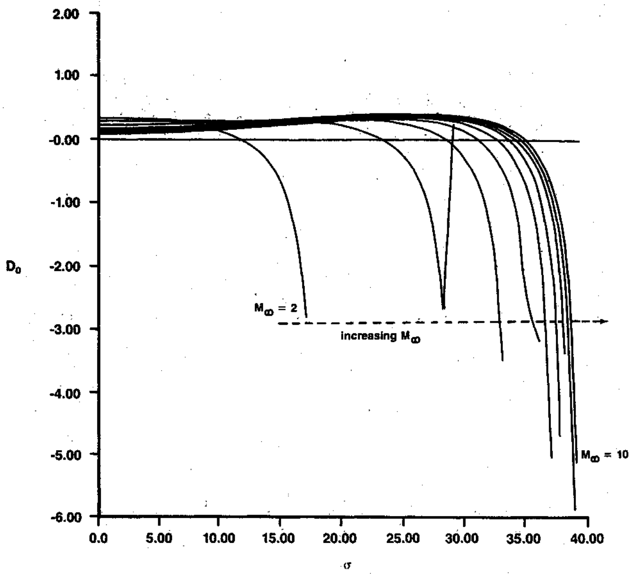

Fig. 5a D_0 vs σ for $M_\infty = 2,3,\ldots,10$.

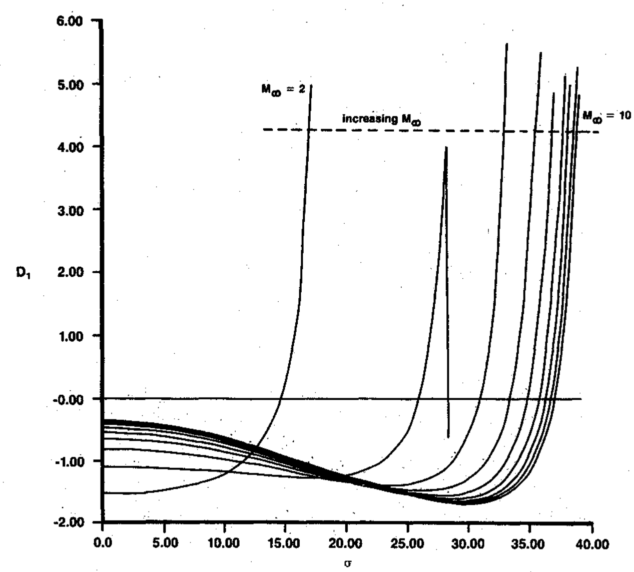

Fig. 5b D_1 vs σ for $M_\infty = 2,3,\ldots,10$.

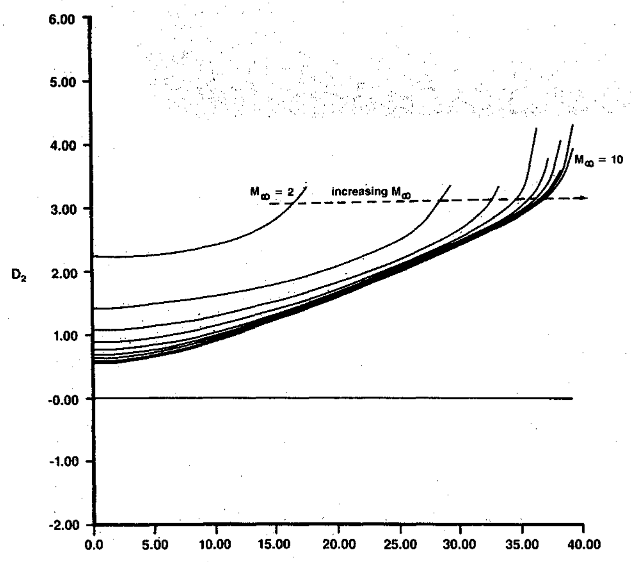

Fig. 5c D_2 vs σ for $M_\infty = 2,3,\ldots,10$.

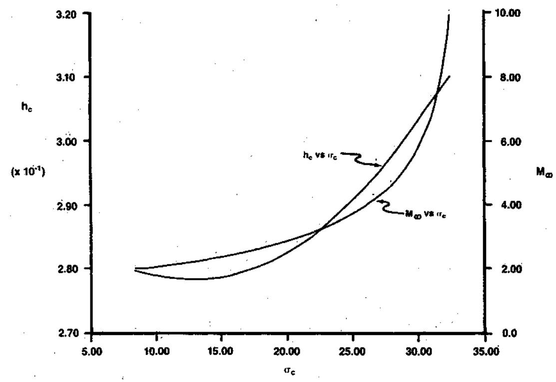

Fig. 6 h_c vs σ_c superimposed on M_∞ vs σ_c.

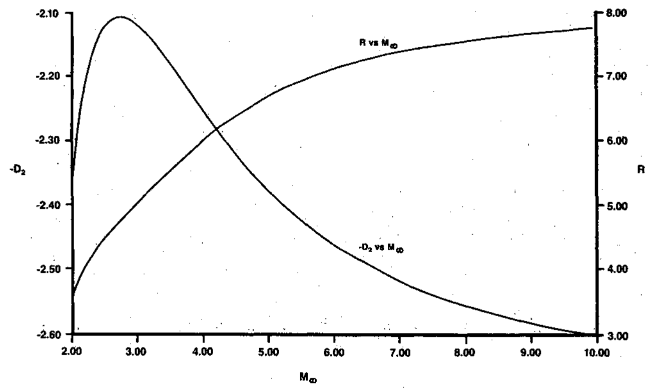

Fig. 7 R vs M_∞ superimposed on $-D_2$ vs M_∞.

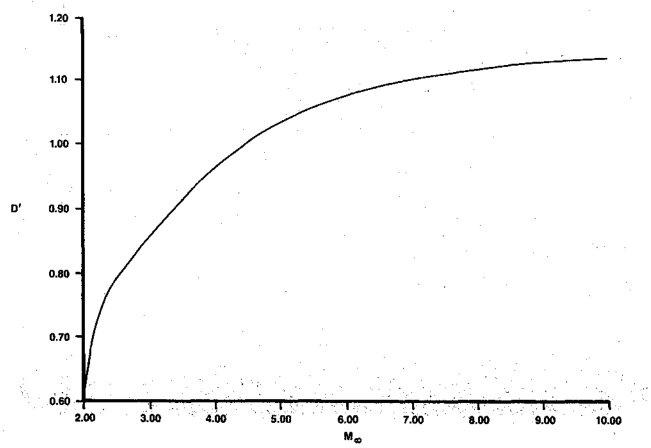

Fig. 8 D' vs M_∞.

for a specific Mach number. In Fig. 7 the relationship between the bifurcating coefficient R and M_∞ is superimposed on the relationship between the second derivative of the real part of the eigenvalue with respect to h (i.e., $-D_2$), and M_∞. With the help of these figures the bifurcating path given by Eq. (15) can be obtained. The relationship between the unfolding parameter $2R\beta/\bar{\sigma} = (\partial D/\partial \bar{\sigma})(\sigma_c, h_c)$ and M_∞ is plotted in Fig. 8. Therefore, Figs. 6 to 8 contain all the information required to completely determine the various bifurcations that can take place.

VI. Conclusion

In this paper, the aerodynamic stability and bifurcation of an aerofoil subject to a single degree of freedom pitching motion has been studied. It was found that, in addition to the simple Hopf bifurcation, degenerate Hopf bifurcation can take place if more than one parameter is allowed to vary. Furthermore, it was shown that in the degenerate case, there will be two periodic bifurcating paths (on both sides of the a axis), with two different frequencies. These frequencies are either both stable or both unstable, as opposed to Hopf bifurcation where the bifurcating path exists either for $\eta > 0$ or for $\eta < 0$. However, the situation giving rise to degenerate Hopf bifurcation is nongeneric. By the introduction of an unfolding parameter, the possible generic bifurcations that can take place near the singularity were obtained. This reveals that for $D_2(\sigma_c)/R > 0$, there exist either two subcritical bifurcations or no bifurcation in a neighborhood of the degeneracy depending upon the sign of β. Similarly, it was found that for $D_2(\sigma_c)/R < 0$ there exist either two supercritical bifurcations or no real solutions in a neighborhood of the degeneracy depending upon the sign of β. In addition, numerical results of the various components of the stiffness and damping derivatives, and other quantities required for the bifurcation analysis, were presented for a thin aerofoil.

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